Math 535: Topology
Homework 1

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Problem 1

Find all topologies on the set $X = \{0, 1, 2\}$.

Solution:
In the list below, $a, b, c \in X$ and it is assumed that they are distinct from one another.

- $\mathcal{P}(X)$ the power set of $X$ (discrete topology).
- $\{\phi, X\}$ (the trivial topology)
- $\{\phi, X, \{a\}\}, a \in \{0, 1, 2\}$. There are 3 of these.
- $\{\phi, X, \{a, b\}\}$ and other combinations of single subsets of size two from $X$. There will be 3 of these.
- $\{\phi, X, \{a, b\}, \{d\}\}, d \in X$. $d$ need not be distinct from $a$ or $b$. There will be 9 of these (3 ways of choosing the subset of size two, and three of choosing the singleton).
- $\{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$. There will be 3 of these (the subsets of size two are determined by the element $a$).
- $\{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. There will be 3 of these.

Problem 2

(i) Show that if $X$ is a set with the trivial topology and $Y$ is any topological space, then every function $f : Y \to X$ is continuous.

(ii) Show that if $X$ is a set with the discrete topology, and $Y$ is any topological space, then every function $f : X \to Y$ is continuous.

Solution:
(i) Let $U \subseteq X$ be open. As $X$ has the trivial topology, this means that $U$ is either $\phi$ or $X$. If $U = \phi$, then $f^{-1}(U) = \phi$, which is open. If $U = X$, then $f^{-1}(U) = Y$, and $Y$ is always open, as it is a member of any topology defined on $Y$. Thus, if $U$ is any open set in $X$, $f^{-1}(U)$ is also an open set in $Y$. Hence, $f$ is continuous.

(ii) Let $U$ be any open set in $Y$. Clearly, $f^{-1}(U)$ is a subset of $X$. However, as $X$ has the discrete topology, all its subsets are open. Hence, $f^{-1}(U) \in \mathcal{P}(X)$ is open in $X$ for any open $U$ in $Y$. Thus, $f$ is continuous.
Problem 3

Use de Morgan’s Laws to prove:

(i) The union of finitely many closed subsets of a topological space is closed.
(ii) The intersection of arbitrarily many closed subsets of a topological space is closed.

Solution:

Let $X$ be the topological space.

(i) Let $A_1, A_2, \ldots, A_n$ be closed sets with $A_i \subseteq X$. Let $B_i = X \setminus A_i$ for each $i$. By the definition of closed sets, all the $B_i$ are open. Now $\bigcup_{i=1}^{n} A_i = X \setminus \bigcap_{i=1}^{n} (X \setminus A_i) = X \setminus \bigcap_{i=1}^{n} B_i$

Since $B_i$ are open, any finite number of intersections of $B_i$ is open. So $\bigcap_{i=1}^{n} B_i = C$ is open. Thus, $\bigcup_{i=1}^{n} A_i = X \setminus C$. As $C$ is open, the finite union of closed sets is closed.

(ii) Let $A_d, \ d \in D$ be arbitrarily many closed subsets of $X$. Let $B_d = X \setminus A_d$. As in the previous section, $B_d$ is open for all $d \in D$. Now $\bigcap_{d \in D} A_d = X \setminus \bigcup_{d \in D} (X \setminus A_d) = X \setminus \bigcup_{d \in D} B_d$.

Since the $B_d$ are open, any union of $B_d$ is also union. So $\bigcup_{d \in D} B_d = C$ is open. Thus, $\bigcap_{d \in D} A_d = X \setminus C$. As $C$ is open, the intersection of arbitrarily many closed sets is closed.

Problem 4

Show that if $X$ is a set and a collection $\sigma$ of subsets of $X$ satisfies:

C1 The set $\sigma$ is closed under finite unions.

C2 The set $\sigma$ is closed under arbitrarily many intersections.

C2 The set $\sigma$ contains $\phi$ and $X$.

Then the collection $\tau = \{U \subseteq X | X \setminus U \in \sigma\}$ is a topology on $X$.

Solution:

1. Let $A_d \in \tau$ with $d \in D$. Note that $\bigcup_{d \in D} A_d = X \setminus \bigcap_{d \in D} (X \setminus A_d)$. As $X \setminus A_d \in \sigma$, it is closed under intersections (C2). Thus, $\bigcap_{d \in D} (X \setminus A_d) \in \sigma$ and $\bigcup_{d \in D} A_d \in \tau$ by the manner in which $\tau$ is constructed.

2. Let $A_1, A_2, \ldots, A_n \in \tau$. Note that $\bigcap_{i=1}^{n} A_i = X \setminus \bigcup_{i=1}^{n} (X \setminus A_i)$. As $X \setminus A_i \in \sigma$, it is closed under finite unions (C1). Thus, $\bigcup_{i=1}^{n} (X \setminus A_i) \in \sigma$ and $\bigcap_{i=1}^{n} A_i \in \tau$ by the manner in which $\tau$ is constructed.

3. $\phi$ is in $\tau$ as $\phi = X \setminus X$ and $X \in \sigma$. Likewise, $X$ is in $\tau$ as $X = X \setminus \phi$ and $\phi \in \sigma$.

Thus, $\tau$ is a topology on $X$. 

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Problem 5

Give an example of a topological space and a collection \( \{W_\alpha\}_{\alpha \in A} \) of closed subsets such that their union \( \bigcup_{\alpha \in A} W_\alpha \) is not closed.

Solution:
Let \( \mathbb{R} \) be the space with the usual topology (i.e. based on the usual metric). Let \( A_n = (-\frac{1}{n}, \frac{1}{n}), \ n \in \mathbb{N} \). Define \( W_n = \mathbb{R} \setminus A_n \). As \( A_n \) is open, \( W_n \) is closed. Now \( \bigcup_{n \in \mathbb{N}} W_n = \mathbb{R} \setminus \bigcap_{n \in \mathbb{N}} A_n \). But this is just \( \mathbb{R} \setminus \{0\} \), which is open as it is the union of two open intervals: \((-\infty, 0) \cup (0, \infty)\). Therefore, this union of closed sets is not closed.

Problem 6

Let \( \mathbb{Q} \subseteq \mathbb{R} \) be the subset of rational numbers. Show that \( \mathbb{Q} \) is neither open nor closed.

Solution: Note that between any two rationals, there exists an irrational. Likewise, between any two irrationals, there exists a rational.

Let \( x \in \mathbb{Q} \). Then every open ball (i.e. interval) around \( x \) necessarily contains an irrational. For example, if \( x \in (a, b) \) where \( a \) and \( b \) are rational numbers, we know that an irrational \( r \) exists with \( a < r < b \) — regardless of the values of \( a \) and \( b \). Therefore, since one cannot find any open intervals about \( x \in \mathbb{Q} \) that are subsets of \( \mathbb{Q} \), \( \mathbb{Q} \) cannot be open.

Likewise, consider \( A = \mathbb{R} \setminus \mathbb{Q} \). This is the set of irrationals. Let \( x \in A \). Again, using the same argument above, let \( x \in (a, b) \) with \( a, b \in A \). There exists a rational \( y \) such that \( a < y < b \). Therefore, there is no open interval about \( x \) that is contained in \( A \). Hence, \( A \) is not open, and therefore \( \mathbb{Q} = \mathbb{R} \setminus A \) is not closed.

Problem 7

Let \( X \) be any set and define a nonempty subset \( U \subseteq X \) to be open if \( X \setminus U \) is finite. Show that this defines a topology on \( X \).

Solution:
(i) Let \( A_d, \ d \in D \) be open subsets of \( X \). Thus, \( B_d = X \setminus A_d \) is finite. \( \bigcup_{d \in D} A_d = X \setminus \bigcap_{d \in D} B_d \). Since \( \bigcap_{d \in D} B_d \subseteq B_d \) and since \( B_d \) is finite, therefore \( \bigcap_{d \in D} B_d \) is finite. Thus, \( \bigcup_{d \in D} A_d = X \setminus C \) where \( C = \bigcap_{d \in D} B_d \) is finite and is therefore an open set as well. Hence, open sets are closed under arbitrary unions.

(ii) Let \( A_1, A_2, \ldots, A_n \) be open subsets of \( X \). Thus, \( B_i = X \setminus A_i \) is finite. Now \( \bigcap_{i=1}^n A_i = X \setminus \bigcup_{i=1}^n B_i \). However, \( \bigcup_{i=1}^n B_i \) is the finite union of finite sets, and hence is itself finite. So
\[ \bigcap_{i=1}^{n} A_i = X \setminus C \text{ where } C = \bigcup_{i=1}^{n} B_i \text{ is finite and is thus an open set.} \]

(iii) Assume \( X \) is nonempty. If \( X \) is finite, then \( X = X \setminus \phi \), and so \( X \) is open as well (\( \phi \) is trivially finite) and is in the topology. Likewise, \( \phi = X \setminus X \), and since \( X \) is finite, \( \phi \) is open and in the topology.

If \( X \) is infinite, pick two distinct elements \( a \) and \( b \) from \( X \). Now \( A = X \setminus \{a\} \) and \( B = X \setminus \{b\} \) are open sets, by definition. As \( X = A \cup B \), \( X \) is also open. Also, \( \phi = A \cap B \), so \( \phi \) is also open.

So regardless of whether \( X \) is finite or infinite, \( \phi \) and \( X \) will be in the topology.

**Problem 8**

Let \( B \) be a basis for a topology on \( X \) and define a subset \( U \subseteq X \) to be open, if for every \( x \in U \), there is a \( V \in B \) such that \( x \in V \subseteq U \). Show that this satisfies the definition of a topology.

**Solution:**

(i) Let \( A_d, d \in D \) be open subsets of \( X \). Let \( E = \bigcup_{d \in D} A_d \). Let \( x \in E \). There must exist a \( A_d \) such that \( x \in A_d \). Since \( A_d \) is open, there exists a \( V_x \in B \) such that \( x \in V_x \subseteq A_d \). However, this means that \( x \in V_x \subseteq E \) as \( A_d \subseteq E \). Thus, by definition, \( E \) is also open. Therefore this collection of open sets is closed under arbitrary unions.

(ii) First, note that if \( V_1, V_2, \ldots, V_n \in B \), and there exists an \( x \) such that \( x \in V_i \forall i \in \{1, 2, \ldots, n\} \), then there exists a \( V \in B \) such that \( x \in V \subseteq \bigcap_{i=1}^{n} V_i \). This follows from a straightforward induction on the definition of a basis.

Now let \( A_1, A_2, \ldots, A_n \) be open sets in \( X \). Let \( x \in E = \bigcap_{i=1}^{n} A_i \). Thus \( x \in A_i \forall i \). Thus, for each \( i \), there exists a \( V_i \) such that \( x \in V_i \subseteq A_i \). Therefore, \( x \in \bigcap_{i=1}^{n} V_i \). By the note in the previous paragraph, there is a \( V \in B \) such that \( x \in V \subseteq \bigcap_{i=1}^{n} V_i \). Since \( \bigcap_{i=1}^{n} V_i \subseteq E \), we have \( V \subseteq E \). Thus, \( E \) is open. Therefore open sets in \( X \) are closed under finite intersections.

(iii) \( \phi \) is open trivially, as there is no \( x \) in \( \phi \). By the definition of a basis, if \( x \in X \), there is a \( V \in B \) such that \( x \in V \). As \( V \subseteq X \), this means that \( X \) is also open.